Observables of Angular Momentum as Observables on the Fedosov Quantized Sphere

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Abstract

In this paper we construct quantum mechanical observables of a single free particle that lives on the surface of the two-sphere $\mathbb{S}^2$ by implementing the Fedosov $*$-formalism. The Fedosov $*$ is a generalization of the Moyal star product on an arbitrary symplectic manifold. After their construction we show that they obey the standard angular momentum commutation relations in ordinary nonrelativistic quantum mechanics. The purpose of this paper is three-fold. One is to find an exact, non-perturbative solution of these observables. The other is to verify that the commutation relations of these observables correspond to angular momentum commutation relations. The last is to show a more general computation of the observables in Fedosov $*$-formalism; essentially an undeformation of Fedosov's algorithm.

1 Introduction

The Moyal star product formalism is an equivalent way to do quantum mechanics.[3] The idea is that instead of using abstract linear operators on a Hilbert space such as position $\hat{x}$ and momentum $\hat{p}$, we may use classical variables x and p however we change the product so that the commutation relations are the same as in the Hilbert space formalism. Namely:

$$[\hat{x}^a, \hat{p}_b] = i\hbar\delta^a_b \quad , \quad \left[\hat{x}^a, \hat{x}^b\right] = 0 = [\hat{p}_a, \hat{p}_b]$$

become:

$$[x^a, p_b]_* = i\hbar\delta^a_b \quad , \quad \left[x^a, x^b\right]_* = 0 = [p_a, p_b]_*$$

we use the convention that the lower case indices run from $1, \ldots, n$ and capital ones run from $1, \ldots 2n$ and:

$$[f, g]_* = f * g - g * f$$

where f and g are any 2 functions of x and p.

We note that the limit $\hbar \to 0^+$ gives the ordinary product of functions.

The definition of the Moyal star for $\mathbb{R}^{2n}$ explicitly is:

$$f * g = f e^{\frac{i\hbar}{2}\omega^{AB}\overleftarrow{\partial}_A \overrightarrow{\partial}_B} g = fg + \frac{i\hbar}{2}\omega^{AB}\left(\partial_A f\right)\left(\partial_B g\right) - \frac{\hbar^2}{8}\omega^{CE}\omega^{AB}\left(\partial_C \partial_A f\right)\left(\partial_E \partial_B g\right) + \cdots$$

where $\partial_A = \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial p_a}\right)$ and the arrow determines the direction that the derivative acts and the operator $\omega^{AB}\overleftarrow{\partial}_A \overrightarrow{\partial}_B$ is called the Poisson bracket.

There is an invertible map called the Weyl transform $\mathcal{W}$ that translates from the Hilbert space formalism to the Moyal formalism. The main property of this transform is that an arbitrary Taylor series operator on the Hilbert space:[1]

$$\hat{A} = \sum_{m,n} A_{a_1 \cdots a_m}{}^{b_1 \cdots b_n} \hat{x}^{a_1} \cdots \hat{x}^{a_m} \hat{p}_{b_1} \cdots \hat{p}_{b_n}$$

becomes by applying the Weyl transform:

$$\mathcal{W}\left(\hat{A}\right) = A = \sum_{m,n} A_{a_1 \cdots a_m}{}^{b_1 \cdots b_n} x^{a_1} * \cdots * x^{a_m} * p_{b_1} * \cdots * p_{b_n}$$

in a mechanical way by simply replacing each $\hat{x}$ with x, $\hat{p}$ with p and placing stars between each of them as is done above.[3]

The trace over an operator of compact support goes to:

$$Tr\left(\hat{A}\right) \overset{\mathcal{W}}{\leftrightarrow} Tr_*\left(A\right) := \frac{1}{\left(2\pi\hbar\right)^n} \int \frac{\omega^n}{n!} A$$

So if we are given the Hamiltonian $\hat{H}$ and the density matrix $\hat{\rho}$ we may map:

$$\hat{H} \overset{\mathcal{W}}{\leftrightarrow} H \quad , \quad \hat{\rho} \overset{\mathcal{W}}{\leftrightarrow} \rho$$

We thus can get the time-independent Schrödinger equation by mapping:

$$\hat{H}\hat{\rho}_n = E_n \hat{\rho}_n \quad , \quad \left[\hat{H}, \hat{\rho}_n\right] = 0$$

to:

$$H * \rho_n = E_n \rho_n \quad , \quad \left[H, \rho_n\right]_* = 0$$

where ρ_n are called the Wigner functions. This also works with the time-dependent Schrödinger equation.[2]

[1] Note that this is effectively an arbitrary operator since we can use the commutators to rearrange each term so that the x's are to the left and the p's are to the right.

[2] See Fedosov for clarification.[1]

Also expectation values become:

$$Tr\left(\hat{\rho}\hat{A}\right) \leftrightarrow Tr_*\left(\rho * A\right)$$

The Moyal $*$ has been generalized to an arbitrary smooth symplectic manifold $(\mathcal{N}, \omega, D)$ endowed with a preserved two-form ω (called the symplectic form) and a phase-space connection D by Fedosov.[1](an excellent summary is [2]) For any such manifold $(\mathcal{N}, \omega, D)$ he gives a perturbative expansion for his $*$-product. However, the convergence issues of the Fedosov $*$, in general, remain unknown.

The properties of the Fedosov $*$ are:

- It is an associative (but not commutative) map $* : C^\infty(\mathcal{N}) \times C^\infty(\mathcal{N}) \to C^\infty(\mathcal{N})$.
- Invariant under <u>all</u> smooth coordinate transformations of the phase-space variables x and p.
- No assumed Hamiltonian.
- The Fedosov $*$ is given perturbatively given any symplectic manifold $(\mathcal{N}, \omega, D)$.
- In the limit $\hbar \to 0^+$, $*$ becomes the ordinary pointwise multiplication of functions on $\mathcal{N}$.
- To first order in $\hbar$ the commutator is the Poisson bracket: $[f, g]_* = i\hbar\{f, g\} + \mathcal{O}\left(\hbar^2\right)$.
- When $\mathcal{N} = T^*\mathbb{E}^n$ (i.e. the phase space or the cotangent bundle of $\mathbb{E}^n$)[3] we get the Moyal $*$.

In this paper we restrict $\mathcal{N}$ to be the cotangent bundle of a manifold with metric g $(\mathcal{M}, g)$ denoted $T^*\mathcal{M}$.[4] The reason to do this is that the cotangent bundle of a manifold is the phase-space of that manifold (i.e. the space of all coordinates x and momentum p). In quantum mechanics using the Moyal $*$ the phase-space is the arena for quantization by giving proper $*$-commutation relations between the x's and p's. The importance of the Fedosov $*$-formalism is that it is a coordinate invariant way of constructing these commutation relations on general $T^*\mathcal{M}$ in such a way that they patch consistently to any coordinate map of the cotangent bundle. Also another important point is that it can be constructed at least perturbatively for any cotangent bundle.

However unlike Fedosov who defines a formulation based on the deformation of covectors (i.e. covectors equipped with a Moyal-like product between them) we will not. We will introduce a Heisenberg algebra generated by $\tilde{s}$ and $\tilde{k}$ ($\left[\tilde{s}^i, \tilde{s}^j\right] = \left[\tilde{k}_i, \tilde{k}_j\right] = 0$, $\left[\tilde{s}^i, \tilde{k}_j\right] = i\hbar\delta^i_j$ where i and j run from 1 through $2n$) at every point of our phase-space $T^*\mathcal{M}$. The motivation to do this instead of Fedosov's way is to make a more direct connection between ordinary quantum mechanics involving Heisenberg algebras and the state spaces that the algebra acts on called Hilbert spaces. We then define this algebra to be linear operators on a Hilbert space which, of course, will eventually contain our states. This new construction will still preserve all of the essential properties of the original Fedosov $*$ albeit reformulated so as to apply to different objects. It will be a quantization procedure

[3]Here $\mathbb{E}^n$ stands for Euclidean n-dimensional space.
[4]The cotangent bundle of any manifold is known to be a symplectic manifold.

i.e. a map of the variables on the phase-space x and p to the observables $\hat{x}$ and $\hat{p}$ which are linear operators on the Hilbert space.

The properties of the Fedosov $*$-quantization in our construction are:

- $\hat{x}$ and $\hat{p}$ form an associative but noncommutative algebra.
- The map from $(x, p) \rightarrow (\hat{x}, \hat{p})$ is invariant under <u>all</u> smooth canonical coordinate transformations of the phase-space variables x and p.
- No assumed Hamiltonian.
- We can construct the $\hat{x}$ and $\hat{p}$ perturbatively given any $(T^*\mathcal{M}, \omega, D)$.
- In the limit $\hbar \rightarrow 0^+$, $\hat{x}$ and $\hat{p}$ become x and p respectively i.e. the ordinary variables on $T^*\mathcal{M}$.
- To first order in $\hbar$ the commutator is the Poisson bracket: $\left[\hat{f}, \hat{g}\right] = i\hbar \left\{\hat{f}, \hat{g}\right\} + \mathcal{O}\left(\hbar^2\right)$.
- When $\mathcal{M} = \mathbb{R}^n$ we get the ordinary quantum mechanics.

In the present work we take as our symplectic manifold $T^*\mathbb{S}^2$, the phase space of a single particle on the 2-sphere, $\mathbb{S}^2$. For this space we construct the Fedosov observables non-perturbatively. The advantage of choosing $\mathbb{S}^2$ is that we had suspected previous to the calculation that the commutators are the same as the usual angular momentum commutators in nonrelativistic quantum mechanics. Saying in fact that the theory of angular momentum is the quantization of the two-sphere without the need for it to be embedded in $\mathbb{R}^3$.

1.1 Outline

We will follow the basic scheme of keeping derivations sufficiently general so as to apply to a completely general manifold with metric $(\mathcal{M}, g)$ and then state results from our specific case of the sphere.

In section 2 we introduce the phase-space connection. We introduce the basis of covectors of matrices/operators $\hat{y}^A$ on the cotangent bundle in section 3. In section 4 we attempt to motivate and solve for a new derivation $\hat{D}$. Also we talk a bit about $\hat{D}$'s ambiguities. Moving into section 5 we explicitly compute the quantities $\hat{x}$ and $\hat{p}$. In section 6 we compute the commutators $\left[\hat{x}^a, \hat{x}^b\right]$, $[\hat{x}^a, \hat{p}_b]$ and $[\hat{p}_a, \hat{p}_b]$ using the explicit forms of the operators. Section 7 explains how one would construct states of angular momentum on $T^*\mathbb{S}^2$ by finally introducing the standard Hamiltonian in ordinary nonrelativistic quantum mechanics. Up until this point no Hamiltonian was assumed.

2 The Phase-Space Connection for $T^*\mathbb{S}^2$

Before we begin, we note the use of the convention that the lower case are the indices of $\mathcal{M}$ (these run from $1,\ldots,n$) and capital ones are the indices of the phase-space $T^*\mathcal{M}$ (these run from $1,\ldots,2n$).

We start with the phase space of a single classical particle confined to a general manifold $(\mathcal{M}, g)$. The objects needed are the phase space, $T^*\mathcal{M}$ which is the cotangent bundle of $\mathcal{M}$, an affine connection on the phase space D and the symplectic form ω of $T^*\mathcal{M}$.

A phase-space connection's action on all functions $f(x,p) \in T^*\mathcal{M}$ and a basis of covectors $\Theta^A \in T^*T^*\mathcal{M}$ are:

$$Df = df = \frac{\partial f}{\partial x^a}dx^a + \frac{\partial f}{\partial p_a}dp_a$$

$$D \otimes \Theta^A = \Gamma^A{}_B \otimes \Theta^B = \Gamma^A{}_{BC}\Theta^C \otimes \Theta^B$$

in such a way as to preserve the symplectic form $\omega = dp_a \wedge dx^a$ on $T^*\mathbb{S}^2$ $(D \otimes \omega = 0)$ where $D = \Theta^C D_C$, $D_C\Theta^A = \Gamma^A{}_{BC}\Theta^B$ and $\Gamma^A{}_{BC}$ is the Christoffel symbol in this basis.

Additionally we impose that D be torsion-free $(D^2 f = 0)$ and that it corresponds to the Levi-Civita connection on $\mathcal{M}$ when it acts on functions of x and dx. Of course we extend to vectors and higher tensors by the Leibnitz rule.

In the specific case of $\mathbb{S}^2$ $(T^*\mathbb{S}^2)$ we employ the convention that the lower/upper-case indices be of the embedding space $\mathbb{E}^3$ $(T^*\mathbb{E}^3)$ running from $1,2,3$ $(1,\ldots,6)$ instead of $1,2$ $(1,\ldots,4)$. We note before continuing that the calculation of the Fedosov observables is inherently two space-time dimensional. The third coordinate is merely for convenience. We see this fact manifest itself by the two conditions (e.g. $\underline{x}\cdot\underline{x} = 1$ and $\underline{x}\cdot\underline{p} = 0$) on the three coordinates every step of the way.

The natural objects and quantities on $T^*\mathbb{S}^2$ are:

- The induced $\mathbb{S}^2$ metric g by the $\mathbb{E}^3$ embedding metric δ.
- The induced $T^*\mathbb{S}^2$ symplectic form ω by the $T^*\mathbb{E}^3$ embedding symplectic form.
- Also the equations defining $T^*\mathbb{S}^2$ inside of $T^*\mathbb{E}^3$, $\underline{x}\cdot\underline{x} = \delta_{ab}x^a x^b = 1$ and $\underline{x}\cdot\underline{p} = x^a p_a = 0$.
- A torsion-free phase-space connection $D = \Theta^A D_A$ on $T^*\mathbb{S}^2$ that preserves all of the above conditions along with the symplectic form ω and there subsequent derivatives. In other words:

$$D^l \otimes g = D^l \otimes \omega = D^l\left(\delta_{ab}x^a x^b\right) = D^l\left(x^a p_a\right) = 0$$

for all positive integers l where $g = g_{ab}dx^a \vee dx^b$, $\omega = \omega_{AB}\Theta^A \wedge \Theta^B$, where Θ^A is basis of forms and $\vee, \wedge$ are the symmetric, antisymmetric tensor products respectively that we will omit because it will be clear when we mean the one or the other.

We define a basis of covectors or forms by:

$$\Theta^A = (\theta^a, \alpha_a)$$

where the θ's are the first three Θ's and the α's are the last three Θ's. θ and α are defined to be:

$$\underline{\alpha} := \underline{x} \times d\underline{p}$$

$$\underline{\theta} := \underline{x} \times d\underline{x}$$

The metric on $\mathbb{S}^2$ is:

$$g = \underline{\theta} \cdot \underline{\theta}$$

The phase-space connection we use for $T^*\mathbb{S}^2$ is:

$$D\underline{x} := d\underline{x} = \underline{\theta} \times \underline{x}$$

$$D\underline{p} := d\underline{p} = \underline{\alpha} \times \underline{x} - \underline{p} \times \underline{\theta}$$

$$D \otimes \underline{\theta} = \underline{\theta} \otimes_\times \underline{\theta} \qquad (D\theta)$$

$$D \otimes \underline{\alpha} = \underline{\theta} \otimes_\times \underline{\alpha} - \frac{2}{3} (\underline{\theta} \times \underline{x}) \otimes \left(\underline{p} \cdot \underline{\theta}\right) + \frac{1}{3} \left(\underline{p} \cdot \underline{\theta}\right) \otimes (\underline{\theta} \times \underline{x}) \qquad (D\alpha)$$

And its corresponding curvature:

$$D^2\underline{x} := 0$$

$$D^2\underline{p} := 0$$

$$D^2 \otimes \underline{\theta} = \tilde{\omega} \otimes (\underline{x} \times \underline{\theta}) \qquad (D^2\theta)$$

$$D^2 \otimes \underline{\alpha} = \tilde{\omega} \otimes (\underline{x} \times \underline{\alpha}) + \frac{1}{3} \left(\underline{\alpha} \left(\underline{\theta} \otimes . \, \underline{\theta}\right) - \underline{\theta} \left(\underline{\alpha} \otimes . \, \underline{\theta}\right) - 2\omega \otimes \underline{\theta}\right) \qquad (D^2\alpha)$$

3 Introducing the $\hat{y}$'s

Following Fedosov, we are going to introduce some machinery namely the operators $\hat{y}$'s to calculate the observables on general manifold $\mathcal{M}$. However, unlike Fedosov who defines these $\hat{y}$'s as covectors equipped with a Moyal-like product between them we choose a different starting point. We define the $\hat{y}$'s at fixed point to be a Heisenberg algebra $\left[\hat{y}^A, \hat{y}^B\right] = i\hbar\omega^{AB}$ where ω^{AB} is the inverse of ω_{AB} with $\omega^{AB}\omega_{BC} = \delta^A_C$. More explicitly $\hat{y}$'s are huge (infinite dimensional) matrices that act on a Hilbert space:

$$\hat{y}^A = \begin{pmatrix} y^A_{11}(x,p) & y^A_{12}(x,p) & \cdots \\ y^A_{21}(x,p) & y^A_{22}(x,p) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where for each A, j, and k $y^A_{jk} \in C^\infty(T^*\mathcal{M})$.

To make a connection with a more familiar form of the Heisenberg algebra we use Darboux's theorem. Darboux's theorem says that in the neighborhood of each point of $q \in T^*\mathcal{M}$ there exist $2n$ local coordinates $\left(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{p}_1, \ldots, \tilde{p}_n\right)$[5], called canonical or Darboux coordinates, such that the symplectic form ω may be written by means of these coordinates as $\omega = d\tilde{p}_1 d\tilde{x}^1 + \cdots + d\tilde{p}_n d\tilde{x}^n$. Thus in this coordinate system at q the $\hat{y}$'s are expressed as $2n$ operators $\left(\tilde{s}^1, \ldots, \tilde{s}^n, \tilde{k}_1, \ldots, \tilde{k}_n\right)$ which have the commutators $\left[\tilde{s}^i, \tilde{s}^j\right] = \left[\tilde{k}_i, \tilde{k}_j\right] = 0$, $\left[\tilde{s}^i, \tilde{k}_j\right] = i\hbar\delta^i_j$ where i and j run from 1 through $2n$. And so at each point the $\hat{y}$'s establish a Heisenberg algebra which acts on a Hilbert space.

Important Note: Fedosov actually begins with the $\hat{y}$'s as being an arbitrary basis of ordinary covectors with a Moyal-like product between themselves.[1] We take the point of view that the specific form of the product is irrelevant. All that matters is that we have an algebra with same commutation relations and the action of the connection is same on the $\hat{y}$'s.

Defining Properties of $\hat{y}$:

$$\left[\hat{y}^A, \hat{y}^B\right] = i\hbar\omega^{AB}$$

$$D\hat{y}^A = \Gamma^A{}_B \hat{y}^B = \Gamma^A{}_{BC}\Theta^C \hat{y}^B \quad , \quad \Theta^A = (\theta^a, \alpha_a)$$

The $\hat{y}$'s commute with the set of quantities $\{x, p, dx, dp, g, \omega, \hbar, i\}$ where i is the complex unit.

Note: The action of the phase-space connection on $\hat{y}$ is the same as the one on Θ ($D \otimes \Theta^A = \Gamma^A{}_{BC}\Theta^C \otimes \Theta^B$) and so we regard it as a basis of operator or matrix-valued covectors.[6] This tells us how to parallel transport the Heisenberg algebra (the $\hat{y}$'s) at one point to the Heisenberg algebra of every other point in a consistent way.

Introducing terminology:

In this paper when we say f is a function/form we define it to be a complex Taylor series in its variables[7]. Explicitly:

$$f(u, \ldots, v) = \sum_{l, j\text{'s}} f_{j_1 \cdots j_l} u^{j_1} \cdots v^{j_l} \quad (j\text{'s are powers not indices})$$

where $f_{j_1 \cdots j_l}$ are constants while u and v could be any of the set $\{x, p, dx, dp, \omega, \hbar, i\}$.

So if f is a function/form of some subset or all of the quantities $x, p, dx, dp, \omega, \hbar$ and i it then commutes with the $\hat{y}$'s and will be called a complex-valued function/form. On the contrary a matrix-valued function/form is a complex Taylor series in $\hat{y}$ and possibly some subset or all of the quantities $x, p, dx, dp, \omega, \hbar$ and i.

[5] Note that these $2n$ coordinates and are different from the $2n+2$ embedding coordinates (x^μ, p_μ).

[6] One may be tempted to quantize the manifold by mapping $\left(x^1, x^2, x^3, p_1, p_2, p_3\right)$ to the matrices $\left(\hat{y}^1, \hat{y}^2, \hat{y}^3, \hat{y}^4, \hat{y}^5, \hat{y}^6\right)$, but we want a coordinate independent formalism and, in general, this is not coordinate independent.

[7] The set of all of these type of functions is sometimes called the enveloping algebra of its arguments.

So if $f(x,p,dx,dp,\omega,\hbar,i)$ is a complex-valued function/form it then commutes with the $\hat{y}$'s. More explicitly with the matrix indices written:

$$\left(\hat{y}^A\hat{y}^B\right)_{jk}=\Sigma_l\hat{y}^A_{jl}\hat{y}^B_{lk}$$

$$\left(\left[\hat{y}^A,f\right]\right)_{jk}:=\hat{y}^A_{jk}f-f\hat{y}^A_{jk}=0$$

On the contrary a matrix-valued function/form does not. From now on we will not write the matrix indices explicitly.

The End Goal:

The idea for Fedosov's introduction of the $\hat{y}$'s is to associate to each $f(x,p)\in C^\infty(T^*\mathcal{M})$ a unique observable $\hat{f}(x,p,\hat{y})$:

$$\hat{f}(x,p,\hat{y})=\sum_l f_{A_1\cdots A_l}\hat{y}^{A_1}\cdots\hat{y}^{A_l} \qquad (\hat{f})$$

where $f_{A_1\cdots A_l}$ are some unknown functions of x and p to be determined.

Important Note: Most of the rest of the sections will be dedicated to finding a solution for $\hat{f}$ (i.e. the coefficients functions $f_{A_1\cdots A_l}$) for each $f(x,p)\in C^\infty\left(T^*\mathbb{S}^2\right)$ up to some "reasonable" ambiguity (discussed in sections 4.1 and 5).

3.1 $T^*\mathbb{S}^2$ Explicitly

Specifically for $T^*\mathbb{S}^2$ we have the induced symplectic form ω of $T^*\mathbb{R}^3$ onto $T^*\mathbb{S}^2$ being:

$$\omega=\underline{\alpha}\cdot\underline{\theta}=\left(\delta^a_b-x^ax_b\right)\alpha_a\theta^b$$

We make the convention[8]:

$$\hat{y}^A=\left(s^a,k_a\right)$$

where the s's are the first three $\hat{y}$'s and the k's are the last three $\hat{y}$'s. Using the above formulas we then write the commutation relations:

$$\left[s^a,s^b\right]=0=\left[k_a,k_b\right]\ ,\ \left[s^a,k_b\right]=i\hbar\left(\delta^a_b-x^ax_b\right)$$

We may assume w.l.o.g. that $\underline{x}\cdot\underline{s}=\underline{x}\cdot\underline{k}=0$ because we observe that the only part of s and k that affect the commutators are the parts that are perpendicular to x. The irrelevance of the part of s and k parallel to x stems from the above relations because $\left[x_as^a,k_b\right]=0$ and $\left[s^a,k_bx^b\right]=0$ and so we could always subtract off the part of s and k parallel to x and get the same commutators. Since $\underline{x}\cdot\underline{s}=\underline{x}\cdot\underline{k}=0$ we have four independent operators which is required since (one for each direction on $T^*\mathbb{S}^2$).

[8]Note that the indices go from 1 to $2n+2$ and are different from the $2n$ operators defined above by $\left(\tilde{s}^1,\ldots,\tilde{s}^n,\tilde{k}_1,\ldots,\tilde{k}_n\right)$. The difference between them is the same as the difference between the embedding coordinates $\left(x^1,\ldots,x^{n+1},p_1,\ldots,p_{n+1}\right)$ and $\left(\tilde{x}^1,\ldots,\tilde{x}^n,\tilde{p}_1,\ldots,\tilde{p}_n\right)$.

The action of the connection and curvature acting on $\underline{s}$ & $\underline{k}$ is written down directly from the equations $(D\theta)$, $(D\alpha)$, $\left(D^2\theta\right)$, and $\left(D^2\alpha\right)$:

$$D\underline{s} = \underline{\theta} \times \underline{s}$$

$$D\underline{k} = \underline{\theta} \times \underline{k} - \frac{2}{3}\underline{\theta} \times \underline{x}\left(\underline{p} \cdot \underline{s}\right) + \frac{1}{3}\left(\underline{p} \cdot \underline{\theta}\right)\left(\underline{s} \times \underline{x}\right)$$

$$D^2\underline{s} = \tilde{\omega}\left(\underline{x} \times \underline{s}\right)$$

$$D^2\underline{k} = \tilde{\omega}\left(\underline{x} \times \underline{k}\right) + \frac{1}{3}\left(\underline{\alpha}\left(\underline{s} \cdot \underline{\theta}\right) + \left(\underline{s} \cdot \underline{\alpha}\right)\underline{\theta} - 2\omega\underline{s}\right)$$

4 Constructing the global derivation $\hat{D}$

Following Fedosov, we now introduce a global derivation as a matrix commutator $\hat{D} = \left[\hat{Q}, \cdot\right]$ which is central to constructing the coefficients $f_{A_1 \cdots A_l}$ in equation $\left(\hat{f}\right)$ for each $f(x,p) \in C^\infty\left(T^*\mathcal{M}\right)$. One possible physical motivation for $\hat{D}$ is that in the next section we will require that all observables $\hat{f}$ must satisfy the equation $\left(D - \hat{D}\right)\hat{f}\left(x,p,\hat{y}\right) = 0$. We see that on $\hat{f}$ $\hat{D}$ is an infinitesimal translation matrix operator equivalent to D. We then reason that matrix operators corresponding to infinitesimal translations on the cotangent bundle should exist i.e. $\hat{D}$. The reason that we require that they must exist is because we are constructing the set of *all* physical matrix operators on states and certainly infinitesimal translations are in this set. If this reasoning is correct then the equation $\left(D - \hat{D}\right)\hat{f} = 0$ must be satisfied for all observables $\hat{f}$. Also the case of $T^*\mathbb{R}^n$ may provide some insight since it is the overlap of this formalism and quantum mechanics using the Moyal $*$ (see in Appendix D for the example of $T^*\mathbb{R}^n$).

Define the derivation $\hat{D}$ by the graded commutator[9]:

$$\hat{D} = \left[\hat{Q}, \cdot\right] = \left[\hat{Q}_A\Theta^A, \cdot\right] \qquad (\hat{D})$$

$$\hat{Q}_A = \sum_l Q_{AA_1 \cdots A_l}\hat{y}^{A_1} \cdots \hat{y}^{A_l}$$

where $\Theta^A = \left(\theta^a, \alpha_a\right)$ and $Q_{AA_1 \cdots A_l}$ are complex-valued functions of x and p that need to be determined. We reiterate that complex-valued functions are not matrices hence they commute with the $\hat{y}$'s.

Again following Fedosov, we can partially determine the functions $Q_{AA_1 \cdots A_l}$ by the mysterious

[9]Graded commutators have the property that $\left[\hat{Q}_A\Theta^A, w\right] = \left[\hat{Q}_A, w\right]\Theta^A = \left(\hat{Q}_A w - w\hat{Q}_A\right)\Theta^A$ where w is an l-form with coefficients $w_{A_1 \cdots A_l}$ which are complex-valued functions of the variables x, p and $\hat{y}$.

equation[10]:

$$\left(D-\hat{D}\right)^2\hat{y}^A=0 \tag{cond $\hat{D}$}$$

The physical motivation for this equation is still unclear and may lurk in the work of Fedosov. One reason for the above requirement is that in the next section we want to solve the equation $\left(D-\hat{D}\right)\hat{f}=0$ for $\hat{f}$ and the above is an integrability condition for the solvability of this equation.

We now let $\hat{Q}$ be the sum of 2 parts the first being the solution in the case of $T^*\mathbb{R}^n$ (Christoffels= $\Gamma=0$):

$$\hat{Q}_A\Theta^A=\omega_{AB}\hat{y}^A\Theta^B+r \tag{$\hat{Q}$}$$

where:

$$r=\sum_l r_{AA_1\cdots A_l}\Theta^A\hat{y}^{A_1}\cdots\hat{y}^{A_l}$$

and $r_{AA_1\cdots A_l}$ are complex-valued functions of x and p that need to be determined. In general, we assume that r has terms that are cubic or higher powers in the $\hat{y}$'s (see Appendix B and Fedosov [1] for clarification).

We rewrite the condition $\left(\text{cond } \hat{D}\right)$ as:

$$\left(D-\hat{D}\right)^2\hat{y}^A=\left[\Omega-Dr+\hat{d}r+r^2,\hat{y}^A\right]=0$$

where $\Omega:=\frac{1}{2i\hbar}\omega_{FN}R^F{}_{BCE}\Theta^C\Theta^E\hat{y}^N\hat{y}^B$ is the phase-space curvature ($D^2\otimes\Theta^A=R^A{}_{BCE}\Theta^C\Theta^E\otimes\Theta^B$) as a commutator and $\hat{d}h=\frac{1}{i\hbar}\left[\omega_{AB}\hat{y}^A\Theta^B,h\right]$ where h is a matrix-valued function of $x,\ p,\ dx,\ dp$ and $\hat{y}$ (see Appendix A for the proof).

From now on we let:

$$\Omega-Dr+\hat{d}r+r^2=0 \tag{r}$$

and keep it in the back of our minds that we could add something that commutes with all $\hat{y}$'s to $\Omega-Dr+\hat{d}r+r^2$.

Important: To emphasize the importance of this equation the reader should note that the whole Fedosov $*$-formalism hinges on this r existing. We know solutions exists perturbatively in general (Fedosov has the recursive solution for it [1] [p.144]), however convergence issues still remain unresolved. On a technical note we have found that solving for r to be the hardest point of the computation of the Fedosov observables because of the need for the right ansatz and the nonlinear equation (r) above that it must solve.

Specifically for the case of $T^*\mathbb{S}^2$ the solution for the curvature as a commutator Ω is:

$$\Omega:=\frac{1}{3}\left(\left(\underline{s}\cdot\underline{\alpha}\right)\left(\underline{s}\cdot\underline{\theta}\right)-s^2\omega\right)+\left(\underline{x}\times\underline{k}\right)\cdot\underline{s}\tilde{\omega}$$

[10]Fedosov adds an additional condition that makes his $\hat{D}$ unique from a fixed D being $\hat{d}^{-1}r_0=0$ where $\hat{d}^{-1}$ is what he calls δ^{-1} (an operator used in a de Rham decomposition) and r_0 is the first term in the recursive solution. We regard this choice as being artificial and thus omit it from the paper.

We then verify that it gives the curvature as commutators:

$$[\Omega, \underline{s}] = [-\underline{k} \cdot (\underline{x} \times \underline{s})\, \tilde{\omega}, \underline{s}] = \tilde{\omega}\, (\underline{x} \times \underline{s})$$

$$[\Omega, \underline{k}] = \frac{1}{3}\left(\underline{\alpha}\,(\underline{s} \cdot \underline{\theta}) + (\underline{s} \cdot \underline{\alpha})\,\underline{\theta} - 2\omega \underline{s}\right) + (\underline{x} \times \underline{k})\,\tilde{\omega}$$

To simplify the calculations we set $i\hbar = 1$ which we will eventually put back in the end.

Fedosov at this point would implement an algorithm to construct r perturbatively, however rather than do this we will make an ansatz for r by exploiting the rotational symmetry of the sphere. This will give us an exact solution for r.[11]

Our ansatz for r is:

$$r = r_0 + f\left(s^2\right) \underline{z} \cdot \underline{s}\,(\underline{x} \times \underline{s}) \cdot \underline{\theta} + g\left(s^2\right) \underline{z} \cdot (\underline{x} \times \underline{s})\, \underline{s} \cdot \underline{\theta} + h\left(s^2\right) \underline{s} \cdot \underline{\theta} \qquad \text{(r ansatz)}$$

where $\underline{z} = \underline{p} - \underline{x} \times \underline{k}$ and $r_0 = \frac{1}{3}\left((\underline{k} \cdot \underline{\theta})\, s^2 - \underline{k} \cdot \underline{s}\,(\underline{s} \cdot \underline{\theta})\right)$.

We will now state the results of our calculations because the calculations are just too space consuming and yet at the same time straight forward. Given the formulas for r and Ω and performing lengthy calculations eventually we get:

$$Dr = \left(\frac{1}{9} - \frac{2g}{3} + \frac{f}{3}\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} + f \underline{\alpha} \cdot (\underline{x} \times \underline{s})\,(\underline{x} \times \underline{s}) \cdot \underline{\theta} - g\,(\underline{s} \cdot \underline{\alpha})\,\underline{s} \cdot \underline{\theta}$$

$$\hat{d}r = -\Omega + \left(2f' s^2 + 3f + g\right) \underline{z} \cdot \underline{s}\tilde{\omega} - g\,(\underline{s} \cdot \underline{\alpha})\,\underline{s} \cdot \underline{\theta} + f \underline{\alpha} \cdot (\underline{x} \times \underline{s})\,(\underline{x} \times \underline{s}) \cdot \underline{\theta}$$

$$r^2 = \left(\frac{1}{9} - \frac{2g}{3} + \frac{f}{3}\right) s^2 \underline{p} \cdot \underline{s}\tilde{\omega} + \left(2gf' s^2 + gf - f^2 - \frac{2f}{3} + \frac{g}{3} - \frac{1}{9}\right) s^2 \underline{z} \cdot \underline{s}\tilde{\omega}$$

where $f' = \frac{\partial f}{\partial (s^2)}$ for all functions.

Putting these into the equation (r) we obtain a condition for g:

$$g = \frac{s^2\left(\left(f + \frac{1}{3}\right)^2 - 2f'\right) - 3f}{s^2\left(\left(f + \frac{1}{3}\right) + 2s^2 f'\right) + 1}$$

while f and h are left arbitrary as long as g is well-defined. This is a necessary and sufficient condition for the equation (r) to hold.

We note that $f = -\frac{1}{3}, g = 1$ and $f = -\frac{1}{12}, g = \frac{1}{4}$ are the only solutions where f and g are constant. We will choose to work with the $f = -\frac{1}{3},\ g = 1,\ h = 0$ solution from now on. We choose this solution for the sake of clarity because it turns out to be the easiest to use in the next few sections. However the reader should note that we calculated the commutators for the general solutions for $g,\ f$ and h and obtained the same result for all of them. See section 6 for the exact result of the

[11] On a technical note: we ran the Fedosov algorithm a few times to help us see what form the ansatz should take. Also remember that when we require $\Omega - Dr + \hat{d}r + r^2 = 0$ modulo terms that commute with the $\hat{y}$'s.

commutators for the particular solution $f = -\frac{1}{3},\ g = 1,\ h = 0$ (and hence the solution for the general solutions for $g,\ f$ and h).

The solution for r for $f = -\frac{1}{3},\ g = 1,\ h = 0$ is:

$$r = -\frac{1}{3}\left(\underline{p} \cdot \underline{s}\right)\left(\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta}\right) + \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{s} \cdot \underline{\theta} \qquad \text{(r soln)}$$

4.1 Ambiguities in r

It is worthwhile to note that the condition $\left(\text{cond } \hat{D}\right)$ does not uniquely define $\hat{D}$ given a fixed D.[12] It appears however that the most of the ambiguities in constructing $\hat{D}$ when given a fixed phase space connection D can be absorbed by a basis change (in other words a gauge transformation). It is easy to see this in a Darboux chart because the connection may be expressed as a commutator:

$$\tilde{D}\hat{y}^A = \left[\tilde{Q}, \hat{y}^A\right]$$

where $\tilde{D} = D - \hat{D}$, $\tilde{Q} = Q - \hat{Q}$ and $D = [Q, \cdot]$. The gauge transformation takes the form:

$$\hat{y}^A \to \hat{y}^A_{new} := U\hat{y}^A U^{-1} \quad , \quad \tilde{D}\hat{y}^A \to \tilde{D}_{new}\hat{y}^A_{new} := \left[U\tilde{Q}U^{-1}, U\hat{y}^A U^{-1}\right] = U\left(\tilde{D}\hat{y}^A\right)U^{-1}$$

where U is some invertible function of the x's, p's and $\hat{y}$'s. Thus the physical content of this theory is independent of U because the commutators remain unchanged.

This can be seen as follows:

$$r \to r + r'$$

where r is a solution to the equation (r) and r' is some unknown series:

$$r' = \sum_l r'_{AA_1 \cdots A_l} \Theta^A \hat{y}^{A_1} \cdots \hat{y}^{A_l}$$

Putting $r \to r + r'$ into (r) we obtain:

$$\Omega - D\left(r + r'\right) + \left[\underline{s} \cdot \underline{\alpha} - \underline{k} \cdot \underline{\theta}, \left(r + r'\right)\right] + \left(r + r'\right)^2 = 0$$

modulo the equation (r) to get:

$$-Dr' + \left[\underline{s} \cdot \underline{\alpha} - \underline{k} \cdot \underline{\theta}, r'\right] + \left(r'\right)^2 + \left[r, r'\right] = 0$$

$$\implies \tilde{D}r' - \left(r'\right)^2 = 0$$

[12] Fedosov adds an additional condition that makes his $\hat{D}$ unique from a fixed D being $\hat{d}^{-1}r_0 = 0$ where $\hat{d}^{-1}$ is what he calls δ^{-1} (an operator used in a de Rham decomposition) and r_0 is the first term in the recursive solution. We regard this choice as being artificial and thus omit it from the paper.

This tells us that if r' is of the form:

$$r' = \left(\tilde{D}U\right)U^{-1}$$

for any U which corresponds to a gauge transformation in the enveloping algebra then the resulting $r_{new} = r + r'$ will solve equation (r). In other words once we have one solution we have actually have huge class of equivalent solutions. We suspect this class of equivalent solutions are all of the solutions for a simply-connected manifold.

Note: There is another source of ambiguity namely the ambiguity in the phase-space connection D. Given a connection D we may add to it a tensor $\Delta^{A}{}_{BC}$ where if we lower by $\Delta_{ABC} = \omega_{AE}\Delta^{E}{}_{BC}$ it is symmetric in all three indices. The new connection still preserves the symplectic form ω. Our curvature becomes:

$$(D + \Delta)^2 = D^2 + D(\Delta) + \Delta^2$$

It is unclear what this ambiguity means so we will leave it for a future discussion.

5 Computing $\hat{x}$ and $\hat{p}$

At this point in Fedosov's algorithm we have all the tools in place to associate an observable $\hat{f}$ to every $f \in C^{\infty}(T^*\mathcal{M})$. Following Fedosov we require that every observable $\hat{f}(x, p, \hat{y})$ must satisfy the equation:

$$\left(D - \hat{D}\right)\hat{f}(x, p, \hat{y}) = 0$$

where $f_{A_1 \cdots A_l}$ are some unknown functions of x and p such that:

$$\ell o\left(\hat{f}(x, p, \hat{y})\right) = f(x, p)$$

ℓo (short for leading order in $\hat{y}$ and $\hbar$) picks out the term which has no $\hat{y}$'s and no $\hbar$'s in it. Explicitly:

$$\hat{f}(x, p, \hat{y}) = f(x, p) + \mathcal{O}(\hat{y}, \hbar)$$

where f has no $\hbar$'s in it.

And so the condition to solve (we believe unique up to unitary transformations) for an observable $\hat{f}$ for every $f \in C^{\infty}(T^*\mathcal{M})$ is:

$$\left(D - \hat{D}\right)\hat{f}(x, p, \hat{y}) = 0 \quad , \quad \ell o\left(\hat{f}(x, p, \hat{y})\right) = f(x, p) \qquad \text{(cond } \hat{f}\text{)}$$

If we have determined our D and $\hat{D}$ we can find solutions for the operators $\hat{x}^a$ and $\hat{p}_a$ (i.e. their coefficients $b^a_{A_1 \cdots A_l}$ and $c_{aA_1 \cdots A_l}$):

$$\hat{x}^a = \sum_l b^a_{A_1 \cdots A_l}\hat{y}^{A_1} \cdots \hat{y}^{A_l} \qquad (\hat{x})$$

$$\hat{p}_a = \sum_l c_{aA_1\cdots A_l} \hat{y}^{A_1} \cdots \hat{y}^{A_l} \tag{$\hat{p}$}$$

where $b^a_{A_1\cdots A_l}$ and $c_{aA_1\cdots A_l}$ are complex-valued functions of x and p (which are the coefficients $f_{A_1\cdots A_l}$ in equation $\left(\hat{f}\right)$ where the first terms in the series is $f = b^a = x^a$ or $f = c_a = p_a$ respectively) and will be determined by the equations:

$$\left(D - \hat{D}\right)\hat{x}^a = 0 \quad , \quad \ell o\left(\hat{x}^a\right) = x^a \tag{cond $\hat{x}$}$$

$$\left(D - \hat{D}\right)\hat{p}_a = 0 \quad , \quad \ell o\left(\hat{p}_a\right) = p_a \tag{cond $\hat{p}$}$$

Again see the example in Appendix D for solutions to $\hat{x}$ and $\hat{p}$ in the case of $T^*\mathbb{R}^n$ where $D = d$.

If we invert the equations $(\hat{x})$ and $(\hat{p})$ once we have solved for the coefficients $b^a_{A_1\cdots A_l}$ and $c^a_{A_1\cdots A_l}$ to get $\hat{y}$ as matrix-valued function of $x, p, \hat{x}$ and $\hat{p}$ (i.e. $\hat{y}^A = \hat{y}^A\left(x, p, \hat{x}, \hat{p}\right)$) and then substitute it into the equation for an arbitrary observable $\left(\hat{f}\right)$ and get:

$$\hat{f}\left(\hat{x}, \hat{p}\right) = \sum_{lm} f^{b_1\cdots b_m}_{a_1\cdots a_l} \hat{x}^{a_1} \cdots \hat{x}^{a_l} \hat{p}_{b_1} \cdots \hat{p}_{b_m} \tag{$\hat{f}$ soln}$$

where $f^{b_1\cdots b_m}_{a_1\cdots a_l}$ are constant coefficients.[13]

However, once have our $\hat{x}$ and $\hat{p}$ there is the ambiguity of how to order each variable when you map a function $f\left(x, p\right)$ to $\hat{f}\left(\hat{x}, \hat{p}\right)$. For example does the function $f\left(x, p\right) = x^1 p_1$ go to $\hat{x}^1 \hat{p}_1$, $\hat{p}_1 \hat{x}^1$ or some linear combination of the two? We should expect this in any well defined quantization procedure because such ordering ambiguities arise in quantum mechanics. We will, for now, regard the ordering of each $\hat{f}$ to be undetermined.[14]

5.1 $T^*\mathbb{S}^2$ Explicitly

Fedosov at this point would implement an algorithm to construct $\hat{x}$ and $\hat{p}$ perturbatively[1] [p.146] for our specific case of $T^*\mathbb{S}^2$. We instead try to find exact solutions to them.[15] Specifically for the case of $T^*\mathbb{S}^2$ we have the ansatz for both $\hat{x}$ and $\hat{p}$ as:

$$\underline{\hat{x}} = v\left(s^2\right)\underline{x} + w\left(s^2\right)\underline{x} \times \underline{s} + y\left(s^2\right)\underline{s}$$

$$\underline{\hat{p}} = \left(\underline{z} \cdot \underline{s} t\left(s^2\right) + \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) q\left(s^2\right)\right)\underline{x} + \underline{z} n\left(s^2\right) + \underline{z} \times \underline{x} u\left(s^2\right)$$

with some functions $v,\ w,\ y,\ t,\ q,\ n$ and u to be determined and the requirements that $\ell o\left(\underline{\hat{x}}\right) = \underline{x}$ and $\ell o\left(\underline{\hat{p}}\right) = \underline{p}$.

[13] To prove this act $D - \hat{D}$ on this equation.

[14] Fedosov chooses Weyl ordering.

[15] We, again, ran the Fedosov algorithm a few times to help us see what for the ansatz should take.

The conditions (cond $\hat{x}$) and (cond $\hat{p}$) become the following equations:

$$\begin{aligned} 0 \quad = \quad & \left(D-\hat{D}\right)\underline{\hat{x}} = \left(\left(-2v'\left(s^2+1\right)+w\right)\left(\underline{s}\cdot\underline{\theta}\right) - y\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right)\underline{x} \\ & + \left(\left(-\frac{v}{s^2} - 2w'\left(s^2+1\right) - w\left(1+\frac{1}{s^2}\right)\right)\left(\underline{s}\cdot\underline{\theta}\right) - y\frac{1}{s^2}\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right)\underline{x}\times\underline{s} \\ & + \left(\left(\frac{v}{s^2} + w\frac{1}{s^2}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} + \left(-2y'\left(s^2+1\right) - y\left(1+\frac{1}{s^2}\right)\right)\left(\underline{s}\cdot\underline{\theta}\right)\right)\underline{s} \end{aligned}$$

and:

$$\begin{aligned} 0 \quad = \quad & \left(D-\hat{D}\right)\underline{\hat{p}} = \left(\begin{array}{c} \left(\begin{array}{c} -2\underline{z}\cdot\underline{s}t'\left(s^2+1\right) - \left(\underline{z}\cdot\underline{s}\right)\frac{1}{s^2}t - 2\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)q'\left(s^2+1\right) \\ +\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\left(1-\frac{1}{s^2}\right)q - \left(\underline{z}\cdot\underline{s}\right)\frac{1}{s^2}u + \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\frac{1}{s^2}n \end{array} \right)\left(\underline{s}\cdot\underline{\theta}\right) \\ \left(\begin{array}{c} -\left(\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\left(1+\frac{1}{s^2}\right)\right)t + \left(\underline{z}\cdot\underline{s}\right)\frac{1}{s^2}q \\ +\underline{x}\cdot\left(\underline{z}\times\underline{s}\right)\frac{1}{s^2}u - \left(\underline{z}\cdot\underline{s}\right)\frac{1}{s^2}n \end{array} \right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} \end{array} \right)\underline{x} \\ & + \left(\left(\begin{array}{c} -\underline{z}\cdot\underline{s}t - \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)q + 2\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)n \\ -\left(\underline{z}\cdot\underline{s}\right)u + 2\left(\underline{z}\cdot\underline{s}\right)\left(s^2+1\right)u' - 2\left(\underline{z}\times\underline{x}\right)\cdot\underline{s}\left(s^2+1\right)n' \\ +\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}u \end{array} \right)\left(\underline{s}\cdot\underline{\theta}\right) \right)\frac{1}{s^2}\underline{x}\times\underline{s} \\ & + \left(\left(\begin{array}{c} 2\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)u + \left(\underline{z}\cdot\underline{s}\right)n \\ -2\left(\underline{z}\cdot\underline{s}\right)\left(s^2+1\right)n' - 2\left(\left(\underline{z}\times\underline{x}\right)\cdot\underline{s}\right)\left(s^2+1\right)u' \\ +\left(\underline{z}\cdot\underline{s}t + \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)q - \underline{z}\cdot\left(\underline{x}\times\underline{s}\right)n\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} \end{array} \right)\underline{s}\cdot\underline{\theta} \right)\frac{1}{s^2}\underline{s} \end{aligned}$$

So the conditions that $\tilde{D}\underline{\hat{x}} = 0$ and $\tilde{D}\underline{\hat{p}} = 0$ becomes 6+6 equations because $\left(\underline{s}\cdot\underline{\theta}\right)^2 = 0 = \left(\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta}\right)^2$ and $\left(\underline{s}\cdot\underline{\theta}\right)\left(\underline{x}\times\underline{s}\right)\cdot\underline{\theta} = \tilde{\omega}$ where $\tilde{\omega}_{ab}$ is invertible. We then solve the subsequent differential equations for the functions $v,\ w,\ y,\ t,\ q,\ n$ and u along with requiring that they have the correct term with no $\hat{y}$'s ($\ell o\left(\underline{\hat{x}}\right) = \underline{x}$ and $\ell o\left(\underline{\hat{p}}\right) = \underline{p}$) in the Taylor expansion to obtain the solutions:

$$\underline{\hat{x}} = \left(\underline{x} - \underline{x}\times\underline{s}\right)\left(s^2+1\right)^{-\frac{1}{2}} \qquad (\hat{x} \text{ soln})$$

$$\underline{\hat{p}} = \left(\underline{z}\cdot\left(\underline{x}\times\underline{s}\right)\underline{x} + \underline{z}\right)\left(s^2+1\right)^{\frac{1}{2}} \qquad (\hat{p} \text{ soln})$$

where $\underline{z} = \underline{p} - \underline{x}\times\underline{k}$ with the following conditions holding:

$$\ell o\left(\underline{\hat{x}}\right) = \underline{x} \quad , \quad \ell o\left(\underline{\hat{p}}\right) = \underline{p}$$

$$\underline{\hat{p}}\cdot\underline{\hat{x}} = \underline{\hat{x}}\cdot\underline{\hat{p}} - 2i\hbar = 0 \qquad (\hat{x}\hat{p} \text{ conds})$$

We note at this point that there is not much insight looking at these formulas except for what we get for the commutators in the next section.

6 The Commutators $\left[\hat{x}^a, \hat{x}^b\right], \left[\hat{x}^a, \hat{p}_b\right]$ and $\left[\hat{p}_a, \hat{p}_b\right]$

Once we have $\hat{x}^a$ and $\hat{p}_a$ i.e. the coefficients $b^a_{A_1 \cdots A_l}$ and $c^a_{A_1 \cdots A_l}$ we work out the commutation relations $\left[\hat{x}^a, \hat{x}^b\right], \left[\hat{x}^a, \hat{p}_b\right]$ and $\left[\hat{p}_a, \hat{p}_b\right]$ using the formulas $(\hat{x})$ and $(\hat{p})$ in the previous section in a brute force calculation. Remember that the $*$-commutators is the Poisson bracket on $T^*\mathcal{M}$ to first order in $\hbar$:

$$\left[\hat{f}\left(\hat{x}, \hat{p}\right), \hat{g}\left(\hat{x}, \hat{p}\right)\right] = \hat{h}\left(\hat{x}, \hat{p}\right)$$

$$\left[f_*\left(x, p\right), g_*\left(x, p\right)\right]_* = h_*\left(x, p\right) = i\hbar\left\{f, g\right\}_{\mathcal{M}} + \mathcal{O}\left(\hbar^2\right) \qquad (*\text{-comm})$$

where $\hat{f}$, $\hat{g}$, $\hat{h}$ and f_*, g_*, h_* are fuctions defined by:

$$\hat{f}\left(\hat{x}, \hat{p}\right) = \sum_{lm} f^{b_1 \cdots b_m}_{j a_1 \cdots a_l} \hbar^j \hat{x}^{a_1} \cdots \hat{x}^{a_l} \hat{p}_{b_1} \cdots \hat{p}_{b_m}$$

$$f_*\left(x, p\right) = \sum_{lm} f^{b_1 \cdots b_m}_{j a_1 \cdots a_l} \hbar^j x^{a_1} * \cdots * x^{a_l} * p_{b_1} * \cdots * p_{b_m}$$

where $f^{b_1 \cdots b_m}_{j a_1 \cdots a_l}$ are constants.

These two sets, one of all f_*'s $\left\{f_*\right\}$ and one of all $\hat{f}$'s $\left\{\hat{f}\right\}$ defined above are isomorphic.

6.1 $T^*\mathbb{S}^2$ Explicitly

In our case of $T^*\mathbb{S}^2$ we find:

$$\left[\hat{x}^a, \hat{x}^b\right] = 0$$
$$\left[\hat{x}^a, \hat{p}_b\right] = i\hbar\left(\delta^a_b - \hat{x}^a \hat{x}_b\right)$$
$$\left[\hat{p}_a, \hat{p}_b\right] = 2i\hbar \hat{x}_{[b}\hat{p}_{a]}$$
$$\underline{\hat{x}} \cdot \underline{\hat{x}} = 1,\ \underline{\hat{p}} \cdot \underline{\hat{x}} = \underline{\hat{x}} \cdot \underline{\hat{p}} - 2i\hbar = 0$$

We now define $\underline{\hat{L}}$ because we argue below that it is a more "natural" momentum:

$$\underline{\hat{L}} := -\underline{\hat{p}} \times \underline{\hat{x}} = \underline{\hat{x}} \times \underline{\hat{p}} = \underline{x} \times \underline{z} + \left(\underline{z} \cdot \underline{s}\right) \underline{x} - \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{s}$$

again with the computed conditions:

$$\underline{\hat{L}} \cdot \underline{\hat{x}} = \underline{\hat{x}} \cdot \underline{\hat{L}} = 0 \quad , \quad \underline{\hat{x}} \cdot \underline{\hat{x}} = 1$$
$$\ell o\left(\underline{\hat{L}}\right) = \underline{L} = \underline{x} \times \underline{p}$$

We easily recognize that $\underline{\hat{L}}$ is the more "natural" variable compared to $\underline{\hat{p}}$. This is because $\underline{\hat{p}} \cdot \underline{\hat{x}} = 0$ and $\underline{\hat{x}} \cdot \underline{\hat{p}} = 2i\hbar$ are very "unnatural" conditions since there is no physical reason why it shouldn't be $\underline{\hat{x}} \cdot \underline{\hat{p}} = 0$ and $\underline{\hat{p}} \cdot \underline{\hat{x}} = -2i\hbar$. We could define $\underline{\hat{p}}_{new} = \underline{\hat{p}} + A\underline{\hat{x}}$ where A is an arbitrary constant and obtain the same commutators. On the other hand the symmetry between $\underline{\hat{L}} \cdot \underline{\hat{x}} = \underline{\hat{x}} \cdot \underline{\hat{L}} = 0$ seems to suggest that $\underline{\hat{L}}$ should be the preferred quantity over $\underline{\hat{p}}$. In other words the relevant component of $\underline{\hat{p}}$ is the one perpendicular to $\underline{\hat{x}}$ which is precisely what $\underline{\hat{L}}$ is.

Therefore the part of $\hat{p}$ parallel to $\hat{x}$ is irrelevant:

$$\underline{\hat{x}} = \left(\underline{x} - \underline{x} \times \underline{s}\right)\left(s^2 + 1\right)^{-\frac{1}{2}} \tag{$\hat{x}$ soln}$$

$$\underline{\hat{L}} = \underline{x} \times \underline{z} + \left(\underline{z} \cdot \underline{s}\right) \underline{x} - \underline{z} \cdot \left(\underline{x} \times \underline{s}\right) \underline{s} \tag{$\hat{L}$ soln}$$

where $\underline{z} = \underline{p} - \underline{x} \times \underline{k}$ with conditions:

$$\underline{\hat{L}} \cdot \underline{\hat{x}} = \underline{\hat{x}} \cdot \underline{\hat{L}} = 0 \quad , \quad \underline{\hat{x}} \cdot \underline{\hat{x}} = 1 \tag{$\hat{x}\hat{L}$ conds}$$

Again we note at this point that there is not much insight looking at these formulas except for what we get for the commutators in the remainder of this section.

We compute the commutators:

$$\left[\hat{x}^a, \hat{x}^b\right] = 0 \tag{xx}$$

$$\left[\hat{x}^a, \hat{L}_b\right] = i\hbar \varepsilon^a{}_{bc}\hat{x}^c \tag{xL}$$

$$\left[\hat{L}_a, \hat{L}_b\right] = i\hbar\varepsilon^c{}_{ab}\hat{L}_c \tag{LL}$$

along with:

$$\underline{\hat{x}} \cdot \underline{\hat{x}} = 1,\ \underline{\hat{L}} \cdot \underline{\hat{x}} = \underline{\hat{x}} \cdot \underline{\hat{L}} = 0 \tag{cond xL}$$

Once we know these relations we know the whole algebra of functions since the algebra is associative. And thus we are done!

And so in the case of $T^*\mathbb{S}^2$ a general element $\hat{f}$ (the function $\left(\hat{f}\right)$ we were looking for and the specific form of the solution $\left(\hat{f}\text{ soln}\right)$) in the space of all observables of $\hat{x}$ and $\hat{L}$ is

$$\hat{f}\left(\hat{x}, \hat{L}\right) = \sum_{lm} f^{b_1\cdots b_m}_{a_1\cdots a_l}\hat{x}^{a_1}\cdots\hat{x}^{a_l}\hat{L}_{b_1}\cdots\hat{L}_{b_m}$$

where $f^{b_1\cdots b_m}_{a_1\cdots a_l}$ are constants. This is the enveloping algebra of the operators of angular momentum and position on a Hilbert space.

Clearly we see that the $\hat{L}$'s generate the standard angular momentum algebra and the $\hat{x}$'s transform properly under rotations. However both the $\hat{x}$'s and the $\hat{L}$'s form a constrained version of the standard $\mathbb{R}^3$ Euclidean algebra with invariant constraints given by the last equations.

7 Angular Momentum States

Since we now have the algebra of observables we can ask about Hamiltonians and states. The free single quantum particle Hamiltonian in ordinary quantum mechanics is $\hat{H} = \frac{\hat{p}^2}{2m} = \frac{\hat{p}_r^2}{2m} + \frac{\underline{\hat{L}}\cdot\underline{\hat{L}}}{mr^2}$ where $\hat{p}_r$ is the radial component of momentum and $\underline{\hat{L}}$ is the angular momentum. In other words the natural choice for the Hamiltonian on our $\mathbb{S}^2$ (which we are free to choose) is $\hat{H} = \underline{\hat{L}} \cdot \underline{\hat{L}}$, $r = 1, m = 1$ because it is just the restricted version of the $\mathbb{E}^3$ free particle Hamiltonian onto $\mathbb{S}^2$. We then construct our angular momentum states in the usual way by solving the eigenvalue equation:

$$\hat{H}\left|\phi\right\rangle = E\left|\phi\right\rangle \tag{Schroedinger}$$

where $E \in \mathbb{R}$.

We won't do it because it is standard physics that one is able to do as an undergraduate physics student.

8 Conclusions

We have explicitly constructed an exact non-perturbative solutions to the observables in the Fedosov $*$-formalism on $T^*\mathbb{S}^2$ and showed that they obeyed the angular momentum commutation relations.

In other words we took the phase space of a single classical particle confined to a sphere, quantized it and got the quantum angular momentum algebra (which we expected). This is done by starting with a chosen phase-space connection D and constructing an explicit formula for $\hat{D}$. Via the equation $\left(D-\hat{D}\right)\hat{f}=0$ that defines the algebra i.e. the algebra of all $\hat{f}$'s we then explicitly constructed $\underline{\hat{x}}$ and $\underline{\hat{p}}$ (the operator analogues of $\underline{x}$ and $\underline{p}$) and computed their commutators. We realized (by defining $\underline{\hat{L}}=\underline{\hat{x}}\times\underline{\hat{p}}$) that the enveloping algebra of all $\underline{\hat{x}}$'s and $\underline{\hat{p}}$'s gives the angular momentum algebra.

Subsequently we defined a Hamiltonian $\underline{\hat{L}}\cdot\underline{\hat{L}}$ that would have eigenstates of angular momentum, however we did not explicitly construct it because it is standard physics.

Another main point was that most of the ambiguity given a fixed phase space connection D of the construction of $\hat{D}$, it seemed, stemmed from the freedom of a change of basis ($\hat{f}\rightarrow U\hat{f}U^{-1}$) given by the argument in section 4.1. And finally the matrix form of the $\hat{y}$'s did not change anything from a Moyal-like object as is done in deformation quantization.

We conclude that we would arrive at the same answer given any algebraic object $\hat{y}$ that had the same commutators along with the same action of the connection on them. We then view the Fedosov $*$-formalism as a general algebraic construction and less tied to the deformation aspect of its original formulation. Thus our formulation using Heisenberg algebras and their subsequent representation spaces (Hilbert spaces) makes a more direct connection to the standard formulation of ordinary quantum mechanics.

9 Acknowledgements

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10 Appendix A

We now show that the equation $\left(D-\hat{D}\right)^2\hat{y}^A=0$ is equivalent to $\left[\Omega-Dr+\hat{d}r+r^2,\hat{y}^A\right]=0$:

Proof:

$$\left(D-\hat{D}\right)^2\hat{y}^A=\left(D^2-D\hat{D}-\hat{D}D+\hat{D}^2\right)\hat{y}^A$$

$$\left(D\hat{D}+\hat{D}D\right)\hat{y}^A=\left[D\left(\omega_{AB}\hat{y}^A\Theta^B+r\right),\hat{y}^A\right]=\left[Dr,\hat{y}^A\right]$$

$$
\begin{aligned}
\hat{D}^2\hat{y}^A &= \left[\hat{Q},\left[\hat{Q},\hat{y}^A\right]\right] = \hat{Q}\left(\hat{Q}\hat{y}^A - \hat{y}^A\hat{Q}\right) + \left(\hat{Q}\hat{y}^A - \hat{y}^A\hat{Q}\right)\hat{Q} \\
&= \left[\hat{Q}^2,\hat{y}^A\right]_- = \left[\left(\omega_{AB}\hat{y}^A\Theta^B + r\right)^2,\hat{y}^A\right]_- = \left[\left(\omega_{AB}\hat{y}^A\Theta^B\right)^2 + \left[\omega_{AB}\hat{y}^A\Theta^B, r\right] + r^2,\hat{y}^A\right]_-
\end{aligned}
$$

$$
2\left(\omega_{AB}\hat{y}^A\Theta^B\right)^2 = \left[\omega_{AB}\hat{y}^A\Theta^B,\omega_{CE}\hat{y}^C\Theta^E\right] = \left[\hat{y}^A,\hat{y}^C\right]\omega_{AB}\Theta^B\omega_{CE}\Theta^E = \omega_{AB}\Theta^A\Theta^B
$$

$$
\implies \hat{D}^2\hat{y}^A = \left[\left[\omega_{AB}\hat{y}^A\Theta^B, r\right] + r^2,\hat{y}^A\right]_-
$$

where $[A,B]_- = AB - BA$ for any A and B.

The curvature D^2 acting on Θ^A is:

$$
D^2 \otimes \Theta^A = R_B{}^A \otimes \Theta^B
$$

Thus the curvature D^2 acting on $\hat{y}^A$ is:

$$
D^2\hat{y}^A = R_B{}^A\hat{y}^B
$$

Knowing this we define Ω as the curvature D^2 acting on $\hat{y}^A$ as a commutator, namely:

$$
\frac{1}{i\hbar}\left[\Omega,\hat{y}^A\right] = R_B{}^A\hat{y}^B
$$

we can immediately write a solution for Ω knowing $\left[\hat{y}^A,\hat{y}^B\right] = i\hbar\omega^{AB}$, $\omega^{AB}\omega_{BC} = \delta^A_C$ and using the symmetries of the curvature tensor:

$$
\Omega := -\frac{1}{2}\omega_{AC}R_B{}^A\hat{y}^B\hat{y}^C
$$

Thus we may rewrite the condition $\left(D-\hat{D}\right)^2\hat{y}^A = 0$ as:

$$
\left(D-\hat{D}\right)^2\hat{y}^A = \left[\Omega - Dr + \hat{d}r + r^2,\hat{y}^A\right] = 0
$$

11 Appendix B

Here we present an argument as to why r only has terms that are cubic or higher powers in the $\hat{y}$'s.

Given:

$$
\hat{D} = \left[\hat{Q},\cdot\right] = \left[\hat{Q}_A\Theta^A,\cdot\right]
$$

$$
\hat{Q}_A = \sum_l Q_{AA_1\cdots A_l}\hat{y}^{A_1}\cdots\hat{y}^{A_l}
$$

we require:

$$
\left(D-\hat{D}\right)^2\hat{y}^A = 0
$$

If we let:

$$\hat{Q}_A \Theta^A = \omega_{AB} \hat{y}^A \Theta^B + r$$
$$r = \sum_l r_{AA_1 \cdots A_l} \Theta^A \hat{y}^{A_1} \cdots \hat{y}^{A_l}$$

If we want r to be globally defined for all manifolds we must define it out of non-degenerate tensors namely the metric, the symplectic form and the curvature. This is because Ω is degree 2 in the $\hat{y}$'s (i.e. $\Omega := -\frac{1}{2}\omega_{AC} R_B{}^A \hat{y}^B \hat{y}^C$ has 2 $\hat{y}$'s). The degree is defined by:

$$\deg(a) = (\text{number of } \hat{y}\text{'s}) + 2\,(\text{number of } \hbar\text{'s})$$

A linear r would yield:

$$\underbrace{\Omega}_{2} - \underbrace{Dr}_{1} + \underbrace{\hat{d}r}_{0} + \underbrace{r^2}_{1}$$

and this cannot be zero for $\Omega \neq 0$. This means that r must have a quadratic term in it.

If r is quadratic ($r = \sum_{l=0}^{2} r_{AA_1 \cdots A_l} \Theta^A \hat{y}^{A_1} \cdots \hat{y}^{A_l}$), in general, there is no way to construct the degree 2 coefficient $r_{AA_1A_2}$ out of invariant tensors. Thus we require that r has terms that are cubic or higher powers in the $\hat{y}$'s. Fedosov mentions this fact also.[1]

For a specific manifold there might be an r that is quadratic. The argument above is meant for an r in a general construction for a general manifold and so we give a counterexample in the case when the manifold $\mathcal{M}$ is $\mathbb{E}^n$.

There is always the trivial solution to r:

$$r = -\frac{1}{2}\omega_{CB} \Gamma^C{}_A \hat{y}^A \hat{y}^B$$

where $\Gamma^C{}_A = \Gamma^C{}_{BA} \Theta^B$ are the Christoffel symbols associated to D. One can easily observe that this is a solution knowing $\left[\hat{y}^A, \hat{y}^B\right] = i\hbar\omega^{AB}$, $\omega^{AB}\omega_{BC} = \delta^A_C$ and using the symmetries of the Christoffel symbols. However the Γ's are not necessarily globally defined and if we find an r in one coordinate patch on $T^*\mathcal{M}$ there is no guarantee that it will be well-defined in another. However if $\mathcal{M} = \mathbb{E}^n$ then this is a global r.

12 Appendix C

Useful identities:

$$d\underline{p} = \underline{\alpha} \times \underline{x} - \underline{p} \times \underline{\theta}$$
$$\theta^a \theta^b = \tilde{\omega} \varepsilon^{abc} x_c$$
$$\underline{z} \times \underline{x} = \underline{p} \times \underline{x} - \underline{k}$$
$$\underline{z} = \underline{p} - \underline{x} \times \underline{k}$$

$$\theta^a \theta^b = \theta^{[a}\theta^{b]} = \frac{1}{2}\varepsilon^{abc}\left(\underline{\theta} \times \underline{\theta}\right)_c = \tilde{\omega}\varepsilon^{abc}x_c$$

$$\left(\underline{v} \times \underline{w}\right) \times \underline{u} = \delta_{ab} v^a \underline{w} u^b - \underline{v}\left(\underline{w} \cdot \underline{u}\right)$$

$$\underline{v} \times \left(\underline{w} \times \underline{u}\right) = \delta_{ab} v^a \underline{w} u^b - \left(\underline{v} \cdot \underline{w}\right)\underline{u}$$

for all 3-D vectors assuming nothing about $[v_a, w_b], [v_a, u_b]$ or $[w_a, u_b]$.

$$\left(\underline{v} \cdot \underline{\theta}\right)\left(\underline{x} \times \underline{w}\right) \cdot \underline{\theta} = \tilde{\omega}\left(\underline{v} \cdot \underline{w}\right)$$

for all 3-D vectors assuming $[\theta^a, v_b] = [\theta^a, w_b] = 0$ and assuming nothing about $[v_a, w_b]$.

For two vectors such that $\underline{v} \cdot \underline{x} = \underline{w} \cdot \underline{x} = 0$ we have the identities:

$$\underline{v} \times \underline{w} = \left(\left(\underline{v} \times \underline{w}\right) \cdot \underline{x}\right)\underline{x} \sim \underline{x}$$

$$\underline{z} \cdot \left(\underline{x} \times \underline{s}\right) = \underline{p} \cdot \left(\underline{x} \times \underline{s}\right) - t$$

$$\left[s^2, \left(\underline{x} \times \underline{k}\right) \cdot \underline{s}\right] = 0$$

$$s_a f\left(\underline{k} \cdot \underline{s}\right) = f\left(\underline{k} \cdot \underline{s} + 1\right) s_a$$

$$\left[r_0, \underline{s}\right] = \frac{1}{3}\left(\left(\underline{s} \cdot \underline{\theta}\right)\underline{s} - s^2 \underline{\theta}\right)$$

$$\left[r_0, \left(\underline{s} \cdot \underline{\theta}\right)\right] = 0$$

$$\left[r_0, s^2\right] = 0 = \left[\underline{z} \cdot \underline{s}, s^2\right]$$

$$\left[r_0, \underline{k}\right] = \frac{1}{3}\left(2\underline{s}\left(\underline{k} \cdot \underline{\theta}\right) - \underline{\theta} t - \left(\underline{s} \cdot \underline{\theta}\right)\underline{k}\right)$$

$$\left[r_0, \underline{z}\right] = \frac{1}{3}\left(\left(\underline{s} \cdot \underline{\theta}\right)\underline{x} \times \underline{k} - \underline{\theta} \times \underline{x} t - 2\underline{x} \times \underline{s}\left(\underline{k} \cdot \underline{\theta}\right)\right)$$

$$\tilde{D}\underline{s} = \underline{\theta} \times \underline{s} - \left(1 + \frac{1}{s^2}\right)\left(\underline{s} \cdot \underline{\theta}\right)\underline{s} - \frac{1}{s^2}\left(\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta}\right)\underline{x} \times \underline{s}$$

$$\tilde{D}\underline{x} = D\underline{x} = \underline{\theta} \times \underline{x} = \frac{1}{s^2}\left(\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta}\right)\underline{s} - \left(\underline{s} \cdot \underline{\theta}\right)\underline{x} \times \underline{s}$$

$$\begin{aligned}\tilde{D}\underline{z} &= \underline{\theta} \times \underline{z} + \left(\left(\underline{z} \cdot \underline{s}\right)\left(\underline{s} \cdot \underline{\theta}\right) - \underline{z} \cdot \left(\underline{x} \times \underline{s}\right)\left(\left(\underline{x} \times \underline{s}\right) \cdot \underline{\theta}\right)\right)\frac{1}{s^2}\underline{s} \\ &\quad + 2\underline{z} \cdot \left(\underline{x} \times \underline{s}\right)\left(\underline{s} \cdot \underline{\theta}\right)\frac{1}{s^2}\underline{x} \times \underline{s}\end{aligned}$$

13 Appendix D: $T^*\mathbb{R}^n$

- In the case of $T^*\mathbb{R}^n$ we solve equation (r) above for r when $D\otimes\Theta^A=0$ therefore $D\hat{y}^A=0$ and hence $\Omega=0$ and get the solution $r=0$. This gives us $\hat{D}$ by the formulas $\left(\hat{D}\right)$ and $\left(\hat{Q}\right)$:

$$\hat{D}=\frac{1}{i\hbar}\left[\omega_{AB}\hat{y}^A\Theta^B,\cdot\right]=\frac{1}{i\hbar}\left[\underline{s}\cdot d\underline{p}-\underline{k}\cdot d\underline{x},\cdot\right]=\frac{1}{i\hbar}\left[(\underline{x}+\underline{s})\cdot d\underline{p}-\left(\underline{p}+\underline{k}\right)\cdot d\underline{x},\cdot\right]$$

where s and k are the first n $\hat{y}$'s and the last n $\hat{y}$'s respectively (i.e. $\hat{y}^A=(s^a,k_a)$) also we have $\left[s^a,s^b\right]=0=\left[k_a,k_b\right]$, $\left[s^a,k_b\right]=i\hbar\delta^a_b$ and $Ds^a=0=Dk_a$.

All operators are required to satisfy:

$$\frac{\partial\hat{f}}{\partial x^a}dx^a+\frac{\partial\hat{f}}{\partial p_a}dp_a-\hat{D}\hat{f}=0$$

$$\Longrightarrow\ \frac{\partial\hat{f}}{\partial x^a}dx^a+\frac{\partial\hat{f}}{\partial p_a}dp_a=\frac{1}{i\hbar}\left[(\underline{x}+\underline{s})\cdot d\underline{p}-\left(\underline{p}+\underline{k}\right)\cdot d\underline{x},\hat{f}\right]$$

This equation is the specific case of the equation $\left(\text{cond }\hat{f}\right)$ for $T^*\mathbb{R}^n$ introduced in section 2.4. The above equation tells us that $\hat{f}$ is a function of $\hat{x}^a=x^a+s^a$ and $\hat{p}_a=p_a+k_a$ $\left(\hat{f}=\hat{f}\left(\hat{x},\hat{p}\right)\right)$ which are solutions to the equation $\left(\text{cond }\hat{f}\right)$ i.e. the coefficients $b^a_{A_1\cdots A_l}$ and $c_{aA_1\cdots A_l}$ in the case of $T^*\mathbb{R}^n$ introduced in the section 2.4 when $\ell o\left(\hat{f}\right)=x^a$ and $\ell o\left(\hat{f}\right)=p_a$ respectively. The equation above implies that $\frac{1}{i\hbar}\left[\cdot,\hat{p}_a\right]$ generates the translation on the cotangent bundle in the x^a-direction and $\frac{1}{i\hbar}\left[\hat{x}^a,\cdot\right]$ generates the translation on the cotangent bundle in the p_a-direction on all observables $\hat{f}$. See Fedosov for more details on motivating the need for $\hat{D}$.[1] $\square$

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